

Planet migration and gap formation by tidally-induced shocks.

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ABSTRACT

Gap formation in a gas disk triggered by disk-planet tidal interaction is considered. Density waves launched by the planet are assumed to be damped as a result of their nonlinear evolution leading to shock formation and its subsequent dissipation. As a consequence wave angular momentum is transferred to the disk, leading to evolution of its surface density. Planetary migration is an important ingredient of the theory; effects of the planet-induced surface density perturbations on the migration speed are considered. A gap is assumed to form when a stationary solution for the surface density profile is no longer possible in the frame of reference migrating with the planet. An analytical limit on the planetary mass necessary to open a gap in an inviscid disk is derived. The critical mass turns out to be smaller than mass M_1 for which planetary Hill's radius equals disk scaleheight by a factor of at least $Q^{5/7}$ (Q is the Toomre stability parameter) depending on the strength of the migration feedback. In viscous disks the critical planetary mass could vary from $\sim 0.1M_1$ to M_1 , depending on the disk viscosity. This implies that a gap could be formed by a planet with mass $(1 - 10) M_\oplus$ depending on the disk aspect ratio, viscosity, and planet's location in the nebula.

Subject headings: planets and satellites: general — solar system: formation — (stars:) planetary systems

1. Introduction.

The recent discovery of extrasolar planets on orbits very close to their parent stars (Mayor & Queloz 1995; Marcy et al. 2000; Vogt et al. 2000; Butler 2001) has raised a number of questions about the formation mechanisms of such systems. These close-in planetary companions are all presumed to be gas giants, typically with masses of the order of Jupiter mass $M_J \approx 2 \times 10^{30}$ g. It is very unlikely that such planets were formed at their present locations (Boss 1995): current theories (Mizuno 1980; Bodenheimer & Pollack 1986) predict that giant planets were formed by gas accretion onto massive ($\sim 15 M_\oplus$) rocky core which themselves are the result of accumulation of a large number of icy planetesimals. The most favorable conditions for this process exist beyond the so-called “snow line” (Hayashi 1981; Sasselov & Lecar 2000) which is estimated to lie several AU from the star, far larger than the actual orbital radii of the known extrasolar planets.

The most popular theory explaining this paradox presumes that giant planets were indeed formed far outside their present locations but then migrated inwards as a result of tidal interaction with a gaseous nebula (Ward 1997a). If a planet is fully immersed in a gas disk then the migration process works as follows: planetary gravitational torques produce density waves carrying angular momentum in the surrounding gas disk. For the planet to migrate inwards (outwards) two conditions must be fulfilled: (1) interaction with the outer (inner) part of the disk should be stronger than with the inner (outer) one, and (2), the waves must not return to the planet. In a finite disk the only way to fulfill the second condition is for the waves to dissipate so that their angular momentum gets transferred to the disk flow. It was demonstrated by Ward (1986) that in Keplerian disks conditions are usually such that the planet tends to migrate inwards. The typical timescale of this process, $\sim 10^3$ yr for a Jupiter-mass planet, is very short compared with the lifetime of the nebula itself ($10^6 - 10^7$ yr). This short timescale presents a significant problem for the migration scenario because the migrating planet is likely to drift right into its parent star in very short time.

Various mechanisms to *stop* migration have been proposed: formation of a cavity in the inner disk by magnetospheric activity of the central star (Shu et al. 1994), resonant interaction with another planet in the system (Masset & Snellgrove 2001), spin-orbital interaction with the central star (Trilling et al. 1998), etc. On the other hand, there is a process almost inevitable in the course of giant planet formation — gap formation in the disk near the planet — which can effectively *slow down* planet migration. When the planet is not very massive its gravity cannot strongly affect the disk surface density distribution in its vicinity and strong interaction with the gas disk through Lindblad resonances (Goldreich & Tremaine 1980, hereafter GT80) leads to the rapid migration described above (so called “type I migration”, Ward 1997a). As the planetary mass grows the drift speed increases but at some point the strength of the torque exerted on the surrounding gas becomes so large that planet simply repels the gas and a gap forms. This strongly diminishes the tidal interaction because it is usually dominated by high-order Lindblad resonances which now lie inside the gap. Thus the orbital evolution of the planet becomes tied to the evolution of the disk and it migrates on the viscous timescale of the disk which could be much longer ($10^5 - 10^6$ yr or even longer, depending on the viscosity in the disk). This stage is called “type II migration” and its existence might help to explain the survival of planetary systems in the course of their orbital evolution or at least significantly alleviate this problem (Ward 1997). Gap formation also provides a reason for the existence of a maximum mass of extrasolar giant planets (Nelson et al. 2000).

In view of all this a very important question arises: how massive should a planet be in order to open a gap? The answer depends not only on the conditions in the nebula (surface density, temperature, viscosity) but also on the dissipation mechanism of planet-induced density waves and the mobility of the planet. If the viscosity in the disk is absent and the planet is not drifting there is no mechanism which could oppose tidal torques and a gap is opened by an arbitrarily small perturber. This conclusion changes if planetary migration is taken into account self-consistently because a low-mass planet is able to drift through the gap before gap fully forms (Ward & Hourigan 1989, hereafter WH89). The planet must slow down its migration somehow for the gap to be opened.

In their study WH89 demonstrated that the disk surface density is enhanced in front of the drifting planet and is reduced behind it. This effect *diminishes* the difference between the torques produced by the planet in the disk outside and inside its orbit and acts to *slow down* the migration. For some mass of the planet its steady drift becomes impossible, the planet halts, and gap formation ensues. Thus, in this picture there is a minimum mass for gap opening even in an inviscid disk. Assuming that wave dissipation is a very rapid process and that angular momentum is immediately transferred to the disk WH89 estimated this critical mass to be

$$M_2 \sim \frac{h}{r} M_f, \quad \text{where} \quad M_f = \Sigma h^2. \quad (1)$$

Here Σ is a surface density of the disk, $h \equiv c/\Omega$ is its geometric thickness (here c is the sound speed and Ω is the angular orbital frequency), and r is the distance from the central star. Typical protosolar nebular parameters were summarized by Hayashi (1980) in the form of the minimum mass Solar nebula (MMSN) model:

$$\Sigma_0(r) = 1700 \text{ g cm}^{-3} \left(\frac{r}{1 \text{ AU}} \right)^{-3/2}, \quad c_0(r) = 1.2 \text{ km s}^{-1} \left(\frac{r}{1 \text{ AU}} \right)^{-1/4}. \quad (2)$$

Using (2) we obtain that $M_f \approx 6 \times 10^{26} \text{ g} \approx 0.1 M_\oplus$ and $M_2 \sim 3 \times 10^{25} \text{ g}$ at 1 AU which is about the mass of Mercury. Of course, if the disk possesses nonzero viscosity this mass would increase because viscous diffusion opposes gap formation.

Lin & Papaloizou (1993, hereafter LP93) present a different point of view on the planetary mass required to open a gap. They also assume damping of the density waves to be local but require mass of the planet to be high enough for the waves to be strongly nonlinear from the beginning and shock immediately transferring angular momentum to the fluid. They showed that this minimum mass corresponds to the case when the Hill's radius of the planet $R_H = r(M_p/M_\star)^{1/3}$ (M_p is the mass of the planet, M_\star is the mass of the central star) is comparable to the vertical disk scaleheight yielding

$$M_1 = \frac{2c^3}{3\Omega G} \approx 14 M_\oplus \left(\frac{r}{1 \text{ AU}} \right)^{3/4} \quad (3)$$

as the relevant mass for opening a gap [the numerical estimate in equation (3) is made for MMSN parameters given in equation (2)]. Another way to look at this restriction is to notice that at $M_p \sim M_1$ the pressure gradient in the disk in the vicinity of the planet becomes so high that epicyclic frequency becomes imaginary and Rayleigh's instability develops. Thus, this restriction could be considered as an upper limit on the mass of the planet that does not open a gap.

Values of the gap opening mass estimated by these two approaches differ by a huge factor. Indeed:

$$\frac{M_1}{M_f} = \frac{2\pi}{3} \frac{\kappa c}{\pi G \Sigma} = \frac{2\pi}{3} Q, \quad (4)$$

where Q is the Toomre stability parameter (Binney & Tremaine 1987) and κ is the epicyclic frequency ($\kappa = \Omega$ for Keplerian rotation law); using MMSN parameters given by equation (2) we obtain that $Q \approx 70$ and $h/r \approx 0.04$ at 1 AU meaning that $M_2 \sim (h/r)Q^{-1}M_1 \approx 5 \times 10^{-4}M_1$.

Ward & Hourigan (WH89) assume that the damping mechanism is independent of the planetary mass and essentially linear, such as viscous dissipation (Takeuchi et al. 1996) or radiative damping (Cassen & Woolum 1996). However these mechanisms are probably ineffective in cold, weakly ionized and optically thick systems such as protoplanetary disks (Hawley & Stone 1998; Goodman & Rafikov 2001, hereafter GR01); in this case tides raised by low-mass planets could propagate much further than just a fraction of disk scale length and gap opening requires significantly more massive perturbers (Ward 1997a).

Requiring waves to be strongly nonlinear and damp immediately as a necessary condition for gap formation is also probably too radical. Indeed, density waves produced by the planet can still evolve due to weak nonlinearity and are able to form a weak shock even if the planet is less massive than M_1 (GR01). This mechanism can lead to a gap formation by lower mass planets than LP93 assumed. Clearly to obtain a reliable estimate of the critical planetary mass one must use a realistic wave damping prescription.

Goodman & Rafikov (GR01) have considered nonlinear evolution of the density waves produced by low-mass planets in two-dimensional disks using the shearing sheet approximation and assuming the background surface density and sound speed to be constant. They have found that a shock is formed quite rapidly (depending on M_p/M_1 , typically after travelling several disk scaleheights from the planet but still not immediately) because the radial wavelength of the perturbation constantly decreases while its amplitude increases (as a consequence of angular momentum flux conservation) thus facilitating wavefront breaking. After wave shocks its angular momentum is gradually transferred to the disk fluid leading to surface density evolution (thus violating the constant surface density assumption). Rafikov (2001) generalized this analysis by taking into consideration the effects of spatial variations of the surface density and sound speed in the disk as well as the cylindrical geometry of the problem. Realistic prescription of the global angular momentum dissipation was provided under the condition that the surface density and sound speed vary on scales larger than the wavelength of the perturbation.

In this paper we study the criterion for gap opening by planetary tides by determining under which conditions a steady-state solution for the disk surface density perturbation is no longer possible (method used in WH89). Planet migration is taken into account self-consistently. Tidal perturbations are supposed to be damped nonlocally by weak nonlinearity and their angular momentum is assumed to be transferred to the disk, as described quantitatively in Rafikov (2001). For the gap opening the assumption of varying surface density in Rafikov (2001) is the most important one because it allows us to solve the problem self-consistently (angular momentum transfer depends on how the surface density is distributed radially).

After developing the general analytical apparatus in §2 we first explore the case of an inviscid disk in §3.1 to highlight the most important physical mechanisms relevant for gap formation. Then we generalize consideration to include the disk viscosity in §3.2. Finally, in §4 we discuss our results.

2. Basic equations

We study the surface density evolution of a gas disk driven by planetary gravitational perturbations. In our model a planet of mass M_p moves in a disk on a circular orbit with radius r_p . The background or initial surface density and sound speed $\Sigma_0(r)$ and $c_0(r)$ are assumed to vary independently on scales $\sim r_p$. The values of the surface density, sound speed, orbital frequency, and disk vertical scaleheight at planet's location are Σ_p, c_p, Ω_p , and h_p respectively. We have summarized brief descriptions of and references to definitions of the most important variables we use in Table 1.

We are only interested in the radial evolution of the surface density and neglect azimuthal variations. Then the general time-dependent equations of disk surface density evolution are the continuity and angular momentum equations (Pringle 1981):

$$r \frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r \Sigma v_r) = 0, \quad (5)$$

$$r \frac{\partial}{\partial t}(\Sigma r^2 \Omega) + \frac{\partial}{\partial r}(r \Sigma v_r \Omega r^2) = -\frac{1}{2\pi} \left(\frac{\partial F}{\partial r} + \frac{\partial G}{\partial r} \right), \quad G = -2\pi r^3 \nu \Sigma \frac{d\Omega}{dr}, \quad (6)$$

where Σ is the disk surface density, v_r is a fluid radial velocity, F is the (azimuthally averaged) angular momentum flux transferred to the disk fluid from the planetary tidal perturbations, and G is the usual viscous angular momentum flux (ν is a kinematic viscosity).

From this system one can easily find that

$$v_r = -\frac{1}{2\pi r \Sigma} \left[\frac{\partial}{\partial r}(\Omega r^2) \right]^{-1} \frac{\partial}{\partial r}(G + F). \quad (7)$$

Substituting this result back into equation (5) one obtains a self-consistent equation for the surface density evolution:

$$\frac{\partial \Sigma}{\partial t} = \frac{1}{2\pi r} \frac{\partial}{\partial r} \left\{ \left[\frac{\partial}{\partial r}(\Omega r^2) \right]^{-1} \frac{\partial}{\partial r}(G + F) \right\}. \quad (8)$$

Now we describe separately all the important ingredients entering this system of equations and all its necessary simplifications.

2.1. Angular momentum flux.

Goldreich & Tremaine (GT80) have considered the gravitational interaction of a gas disk with a planet embedded in it. They have demonstrated that a perturber on a circular orbit exerts a torque on the disk only in the immediate vicinity of the Lindblad resonances. Density waves launched at these locations carry angular momentum away from the satellite in the outer disk and towards it in the inner one. Lindblad resonances of order $m \sim r_p/h_p \gg 1$ located about one scaleheight

Table 1. Key to important symbols.¹

Symbol	Meaning	Where defined
Q	Toomre’s Q	equation (4)
α	viscous α -parameter	§2.3
h	disk vertical scaleheight	§1
γ	gas polytropic index	§2.1
F_0	total angular momentum flux	equation (9)
$\mu_{max}(Q)$	function characterizing torque cutoff	equation (9)
$F_J(r)$	distance-dependent angular momentum flux	equation (10)
t	distance-like variable for the shock propagation	equation (12)
$\varphi(t)$	angular momentum damping function	equation (14)
l_p	Mach-1 length	equation (12)
M_1	critical planetary mass for a strong wave nonlinearity	equation (3)
M_2	critical planetary mass for a strong migration feedback	equation (1)
M_f	fiducial mass	equations (1) & (25)
x	radial distance from planet scaled by l_p	equation (16)
x_{sh}	dimensionless shocking distance	equation (21)
ζ	proportionality constant in definition of x_{sh}	equation (21)
z	radial distance from planet scaled by the shocking distance	equation (22)
σ	dimensionless surface density	equation (19)
ρ_Σ	dimensionless surface density at planet’s location	equation (19)
v_d	drift velocity	§2.2 & equation (26)
β	relative difference in torques leading to migration	equations (26) & (B4)
v	correction factor for drift velocity	equation (26)
λ_s	parameter characterizing strength of the feedback	equation (27)
z_0	cutoff distance	equation (27)
t_0	characteristic timescale	equation (29)
λ_ν	viscous parameter	equation (30)
λ_t	tidal parameter	equation (31)

¹Symbols with subscript “p” imply the value of the corresponding quantity at the planet’s location.

$\sim h_p$ from the planet are the strongest contributors to the angular momentum flux: lower order resonances are far from the planet and do not feel such strong tidal forcing while higher order ones are saturated because of the so-called “torque cutoff” (GT80). The total angular momentum flux across a cylinder of radius r_p carried by all density wave harmonics is shown to be given by the expression (GT80)

$$F_0 = (GM_p)^2 \frac{\Sigma_p r_p \Omega_p}{c_p^3} \left\{ \frac{4}{9} \mu_{max}^3(Q) [2K_0(2/3) + K_1(2/3)]^2 \right\}, \quad (9)$$

where K_ν denotes the modified Bessel function of order ν and the function $\mu_{max}(Q)$ describes the strength of the torque cutoff. It depends only on the disk stability parameter Q mentioned previously in equation (4), and $\mu_{max} \approx 0.69$ for disks with $Q \gg 1$.

Beyond several h_p from the planet, tidal perturbations behave basically like sound waves (Lin & Shu 1968). Their nonlinear evolution finally leads to shock formation (GR01). After this happens the angular momentum flux of the density waves decreases because it gets transferred to the disk fluid. Taking this into effect account Rafikov (2001) has demonstrated that the angular momentum flux carried by the tidal perturbations F_J is given by

$$F_J(r) = \frac{2^{3/2} c_p^3 r_p \Sigma_p}{(\gamma + 1)^2 |2A(r_p)|} \Phi(M_p, t), \quad \text{where} \quad (10)$$

$$\Phi(M_p, t) = \int \chi^2(M_p, t, \eta) d\eta = \left(\frac{M_p}{M_1} \right)^2 \Phi \left(M_1, \frac{M_p}{M_1} t \right), \quad (11)$$

$$t(r) = -\frac{r_p}{l_p} \int_{r_p}^r \frac{\Omega(r') - \Omega_p}{c(r') g(r')} dr', \quad g(r) = \frac{2^{1/4}}{r_p c_p \Sigma_p^{1/2}} \left(\frac{r \Sigma c^3}{|\Omega - \Omega_p|} \right)^{1/2}. \quad (12)$$

Here $A = (r/2)d\Omega/dr$ is Oort’s A constant, l_p is the Mach-1 distance [the distance from the planet at which shear velocity is equal to sound speed, $l_p = (2/3)h_p$ for Keplerian disks], γ is a polytropic index, and M_1 is given by equation (3). The variable $t(r)$ plays the role of the distance travelled by the wave from the planet, and the dimensionless flux $\Phi(M_1, t)$ was calculated in GR01 for $M_p = M_1$. In equations (10)-(12) the surface density Σ and sound speed c can vary arbitrarily with radius r .

It was also demonstrated in GR01 that the density wave shocks when t reaches

$$t_{sh} = 0.79 \frac{M_1}{M_p}. \quad (13)$$

In this expression we have neglected an additional term equal to 1.89, which takes into account the finite distance required for the wake formation in the linear regime. One can do this only if $M_p \ll M_1$ (which is always true in this paper), so that the region of linear wake formation is separated from the region of its weakly nonlinear evolution. Otherwise these regions overlap and the wave shocks before it is fully formed, significantly complicating the analysis.

For $t \leq t_{sh}$, the total angular momentum flux of the waves F_J is conserved and has to coincide with F_0 given by the expression (9); the value of $\Phi(t)$ for $t \leq t_{sh}$ is such that this condition is

fulfilled. Note also that although the polytropic index γ enters equation (10) the dimensionless flux F_J does not actually depend on it because $\Phi \propto (\gamma + 1)^2$ (see GR01). Thus, we can rewrite equation (10) as

$$F_J(r) = F_0 \varphi \left(\frac{M_p}{M_1} t(r) \right), \quad \text{where} \\ \varphi(t) = \Phi(M_p, t) / \Phi(M_p, t_{sh}), \quad (14)$$

The function $\varphi(t)$ describes the damping after the shock is formed and is such that $\varphi(t) = 1$ when $t \lesssim t_{sh}$, $d\varphi(t)/dt = 0$ when $t = t_{sh}$, and $\varphi(\infty) = 0$. Its behavior is shown in Fig. 7. In this paper instead of the true angular momentum flux dependence calculated by GR01 we will be using for numerical convenience a simple analytical fit satisfying all of the above mentioned conditions and described in Appendix A.

From the conservation of the total angular momentum (assuming a steady state) it follows that the corresponding angular momentum flux of the disk fluid is

$$F(r) = F_0 \left[1 - \varphi \left(\frac{M_p}{M_1} t(r) \right) \right]. \quad (15)$$

This expression together with equations (10)-(12) is to be substituted into equation (8) and then the surface density evolution could be calculated self-consistently because the dependence of F on Σ is taken explicitly into consideration by equation (12).

2.2. Local approximation.

Using equations (10)-(12) one can in principle study the temporal evolution of Σ taking into account the fact that Σ_0 , c_0 , Ω_0 , etc. vary on scales $\sim r_p$. We prefer to study the problem in a simplified setting assuming that although gap formation is a nonlocal process it still occurs sufficiently close to the planet. Later we will determine the necessary conditions for this assumption to be true.

This simplification allows us to set $c = c_p$, $r = r_p$, and to use the shearing sheet approximation to represent the background fluid flow. The only disk quantity which is assumed to vary in our consideration is the surface density Σ . By setting $c = c_p$ we imply that rapid variations of Σ do not violate the thermal balance in the disk that determines $c(r)$. Here the disk temperature distribution is supposed to be the result of external irradiation by the central star and our assumption of constant c rests upon the absence of geometrical effects such as the shadowing by the gap edge (Lecar & Saslaw 1999; Dullemond et al. 2001).

In this “quasilocal” approximation, introducing a new variable

$$x = \frac{r - r_p}{l_p}, \quad (16)$$

we obtain from equation (12) that

$$t(x) = \text{sign}(x) 2^{-1/4} \int_0^x \left[\frac{\Sigma_p}{\Sigma(x')} \right]^{1/2} |x'|^{3/2} dx'. \quad (17)$$

If $\Sigma(x) = \Sigma_p$ we reproduce the expression for $t(x)$ obtained in GR01.

The disk-planet interaction leads to the radial migration of the planet (GT80; Ward 1997a) with some drift velocity v_d . We consider the disk evolution in the coordinate system comoving with the planet. In this case

$$\Sigma(r, t) \rightarrow \Sigma(r - v_d t, t) \quad \text{and} \quad \frac{\partial \Sigma}{\partial t} \rightarrow \frac{\partial \Sigma}{\partial t} - v_d \frac{\partial \Sigma}{\partial r}. \quad (18)$$

Usually it is found that migration is inward, i.e. $v_d < 0$ (Ward 1986). Using the definitions

$$\sigma(r, t) \equiv \frac{\Sigma(r, t)}{\Sigma(\infty)}, \quad \rho_\Sigma \equiv \frac{\Sigma_p}{\Sigma(\infty)}, \quad (19)$$

one can rewrite equation (8) in the following form:

$$\frac{\partial \sigma}{\partial t} - \frac{v_d}{l_p} \frac{\partial \sigma}{\partial x} = \frac{3\nu}{l_p^2} \frac{\partial^2 \sigma}{\partial x^2} + \frac{F_0}{\pi \Sigma(\infty) \Omega r^2 l_p^2} \frac{\partial^2}{\partial x^2} \left[\varphi \left(\text{sign}(x) \frac{\rho_\Sigma^{1/2}}{2^{1/4}} \frac{M_p}{M_1} \int_0^x \frac{|x'|^{3/2}}{\sigma^{1/2}} dx' \right) \right]. \quad (20)$$

Here $\Sigma(\infty)$ is the surface density of the disk far away from the planet; it is different from Σ_p in the case of viscous disks ($\rho_{ho\Sigma} \neq 1$, see §3.2). Then the ratio ρ_Σ is not known a priori and has to be found from a self-consistent solution of the problem. We take $\Sigma(\infty)$ to be the same both inside and outside of the planet which means that global surface density gradients are neglected here as we have mentioned earlier.

GR01 has demonstrated that in the constant surface density disk, the wave shocks after propagating a distance (in x)

$$x_{sh} \approx 1.4 \left(\frac{\gamma + 1}{12/5} \frac{M_p}{M_1} \right)^{-2/5} = \zeta \left(\frac{M_p}{M_1} \right)^{-2/5}, \quad \zeta = 1.4 \left(\frac{\gamma + 1}{12/5} \right)^{-2/5}. \quad (21)$$

away from the planet. In what follows we will always assume for simplicity that $\zeta = 1.4$ thus taking $\gamma = 7/5$ [see however Fridman & Gor'kavyi (1999)]. In a disk with varying surface density x_{sh} is no longer the distance after which the shock forms. Still, it is convenient to change x to a new spatial coordinate z given by

$$z \equiv \frac{x}{x_{sh}} = \frac{x}{\zeta} \left(\frac{M_p}{M_1} \right)^{2/5}. \quad (22)$$

In terms of z , the surface density in our particular problem varies on scales $z \sim 1$. Thus, the condition for applying our “quasilocal” approach (see §2.2) is

$$1 \lesssim x_{sh} \lesssim r_p/h_p. \quad (23)$$

We check later in §4 if this is fulfilled in realistic protoplanetary systems.

With the aid of (22) equation (20) transforms to

$$\begin{aligned} \frac{\partial \sigma}{\partial t} - \frac{v_d}{\zeta l_p} \left(\frac{M_p}{M_1} \right)^{2/5} \frac{\partial \sigma}{\partial z} &= \frac{3\nu}{\zeta^2 l_p^2} \left(\frac{M_p}{M_1} \right)^{4/5} \frac{\partial^2 \sigma}{\partial z^2} \\ &+ \frac{9}{4} \frac{F_0}{\zeta^2 \pi M_f \Omega r^2} \left(\frac{M_p}{M_1} \right)^{4/5} \frac{\partial^2}{\partial z^2} [\varphi(I(z))], \end{aligned} \quad (24)$$

$$I(z) = \text{sign}(z) \frac{\rho_\Sigma^{1/2} \zeta^{5/2}}{2^{1/4}} \int_0^z \frac{|z'|^{3/2}}{\sigma^{1/2}} dz', \quad M_f = \Sigma(\infty) h^2. \quad (25)$$

Equation (24) does not have M_p inside the argument of φ and this permits one to extract an explicit dependence on the planetary mass.

2.3. Evolution equations.

Equation (24) can only describe the surface density evolution when it is supplied with the information about the behavior of the migration velocity v_d as a function of σ . In Appendix B we demonstrate that

$$v_d = -\frac{2\beta F_0}{M_p r_p \Omega_p} \frac{h_p}{r_p} v, \quad \text{where} \quad v = 1 - \lambda_s \left(\int_{-\infty}^{-z_0} - \int_{z_0}^{\infty} \right) \frac{\rho_\Sigma^{-1} \sigma - 1}{z^4} dz, \quad (26)$$

is a correction factor for the drift velocity caused by the feedback from the surface density variations to the planetary migration; also

$$\lambda_s \equiv \frac{3}{\beta \zeta^3 \mu_{max}^3} \frac{r_p}{h_p} \left(\frac{M_p}{M_1} \right)^{6/5} \quad \text{and} \quad z_0 = (\mu_{max} x_{sh})^{-1}. \quad (27)$$

In the expression for v the migration feedback is represented by second term — it takes into account modification of the contribution from low-order Lindblad resonances, which is important because the planetary torque in the outer and inner parts of the disk due to high-order Lindblad resonances almost completely cancel. This means that even a small contribution from low-order resonances could produce a significant effect on the migration speed. When the feedback is absent the drift speed is determined only by global gradients of Σ_0 and c_0 represented by factor β in equation (26) (see Appendix B).

We will see later that in interesting cases the planet produces variations of $\sigma \sim 1$ and $\sigma > 1$ in the direction of migration and $\sigma < 1$ in the opposite direction: a surface density increase forms in the disk *in front* of the planet, and a depression appears *behind* it. Thus density perturbations on both sides of the planet work in one direction — to enhance the effect of the disk in front of

the drifting planet and decrease it behind the planet [neglecting for a moment the fact that factor ρ_Σ in equation (26) could be different from unity]. They do not cancel each other (like in the case of migration due to variations of Σ_0) and produce $O(1)$ effect. This means that planetary-driven surface density perturbations tend to *stall* migration. However, these variations are strongest at distances $\sim x_{sh}h_p \gg h_p$ from the planet, where torque generation is weak, thus they need to be large to compete with usual drift due to weak global gradients (for which cancellation effects are important).

Using this and the results of previous sections we can rewrite equations (24)-(25) to find the following form of the evolution equation :

$$\rho_\Sigma^{-1} \frac{\partial \sigma}{\partial \tau} + v \frac{\partial \sigma}{\partial z} = \rho_\Sigma^{-1} \lambda_\nu \frac{\partial^2 \sigma}{\partial z^2} + \lambda_t \frac{\partial^2 \sigma}{\partial z^2} [\varphi(I(z))], \quad (28)$$

$$\tau \equiv \frac{t}{t_0}, \quad t_0 \equiv \Omega^{-1} \times \frac{3\zeta}{4\beta} \left\{ \frac{4}{9} \mu_{max}^3(Q) [2K_0(2/3) + K_1(2/3)]^2 \right\}^{-1} \left(\frac{M_1}{M_p} \right)^{7/5} \frac{M_1}{M_f} \frac{r_p}{h_p}, \quad (29)$$

$$\lambda_\nu \equiv \alpha \frac{81}{16\zeta\beta} \left\{ \frac{4}{9} \mu_{max}^3(Q) [2K_0(2/3) + K_1(2/3)]^2 \right\}^{-1} \left(\frac{M_p}{M_1} \right)^{-3/5} \frac{M_1}{M_f} \frac{r_p}{h_p}, \quad (30)$$

$$\lambda_t \equiv \frac{3}{4\zeta\beta} \left(\frac{M_p}{M_1} \right)^{7/5} \frac{M_1}{M_f}. \quad (31)$$

Here we have parameterized viscosity using the usual α -prescription: $\nu = \alpha hc$ and the correction factor for drift velocity v is given by equation (26).

We will often use the numerical form of the parameters entering this system, namely

$$\lambda_t \approx 0.16 \, Q \left(\frac{M_p}{M_1} \right)^{7/5}, \quad \lambda_s \approx 0.48 \, \frac{r_p}{h_p} \left(\frac{M_p}{M_1} \right)^{6/5}, \quad (32)$$

$$\lambda_\nu \approx 1.2 \, \alpha \, Q \, \frac{r_p}{h_p} \left(\frac{M_p}{M_1} \right)^{-3/5}, \quad t_0 = \Omega^{-1} \times 0.3 \, Q \, \frac{r_p}{h_p} \left(\frac{M_p}{M_1} \right)^{-7/5}. \quad (33)$$

where we have used equation (4) and our calculation assumes that $Q \gg 1$, $\beta \approx 7$, and $\zeta \approx 1.4$.

Equation (28) describes the evolution of the disk surface density and its solution depends on 5 dimensionless parameters: $\lambda_\nu, \lambda_t, \lambda_s, t_0$, and z_0 which themselves are combinations of h_p/r_p , M_p/M_1 , α , and Q . The parameter λ_t describes the effect of the planetary tidal torques on the surface density distribution, while λ_ν tells us how strong the viscosity is. Obviously, for the gap to form the second term in the r.h.s. of (28) must be larger than the viscous one. The feedback which planetary migration receives from the surface density perturbations is described by the parameter λ_s defined in equation (27) and it is strongest for large λ_s . Feedback also depends on the value of z_0 [equation (27)] which represents the effect of the torque cutoff.

To understand better the meaning of these parameters let us consider the time Δt_d which it takes for a planet to drift across a gap with the width Δr . Obviously, $\Delta t_d = \Delta r/v_d$, where again

v_d is a drift velocity. Using equations (9) and (26) one can easily obtain that

$$v_d \sim c \frac{\beta}{Q} \frac{h_p}{r_p} \frac{M_p}{M_1}, \quad \text{and} \quad \Delta t_d \sim \frac{\Delta r}{c} \frac{Q}{\beta} \frac{r_p}{h_p} \frac{M_1}{M_p}. \quad (34)$$

Time needed for viscous diffusion to fill the gap is clearly

$$\Delta t_\nu \sim \frac{(\Delta r)^2}{\nu} \sim \frac{1}{\Omega_p} \frac{1}{\alpha} \left(\frac{\Delta r}{h_p} \right)^2. \quad (35)$$

Finally, for the tidal torques to clear out a gap with width Δr an angular momentum $H = \Sigma \Omega (r \Delta r)^2$ must be supplied to the gas (see GT80). If we assume that angular momentum flux carried by the density waves is damped on the characteristic length $\sim \Delta r$, then the timescale for gap opening is

$$\Delta t_{open} \sim \frac{H}{F_0} \sim \frac{1}{\Omega} \frac{r_p}{h_p} \left(\frac{\Delta r}{h_p} \right)^2 \left(\frac{M_1}{M_p} \right)^2. \quad (36)$$

In our particular problem the characteristic gap width and the wave damping distance are the same and given by $\Delta r \sim h_p x_{sh} = \zeta h_p (M_p/M_1)^{-2/5}$ [see equation (21)]. Then, substituting this into equation (34) we obtain that $t_0 \sim \Delta t_d$, i.e. time t_0 is a typical timescale for a gas drift through the gap caused by the planetary migration (in planet's reference frame). One can see that $t_0 \gg \Omega^{-1}$ for $M_p \ll M_1$ which justifies our use of azimuthally averaged quantities if $\lambda_t \sim 1$ [because t_0/λ_t is a characteristic timescale for the gap opening, see equation (28)]. Also for this choice of Δr one can easily find that $\lambda_t \sim \Delta t_d/\Delta t_{open}$, that is drift through the gap cannot replenish material repelled by planetary torques when $\lambda_t \gtrsim 1$, and gap could be opened if viscosity is absent. We will further confirm this conclusion in §3.1. Likewise, one can see that $\lambda_\nu \sim \Delta t_d/\Delta t_\nu$ which means that viscous diffusion is more important for the surface density evolution than effects of the migration when $\lambda_\nu \gtrsim 1$. It is also clear that for tidal torques to overcome viscosity and open a gap in a viscous disk one needs $\Delta t_{open} \lesssim \Delta t_\nu$, or $\lambda_t \gtrsim \lambda_\nu$ (see §3.2).

3. Kinematic wave solution

It is possible that for some values of parameters $\lambda_\nu, \lambda_t, \lambda_s$ solution for the surface density does not depend on time-like variable τ . In this case, the profile of σ is stationary in the reference frame migrating with the planet. Although planetary gravitational torques tend to repel the disk fluid, planet is mobile enough to migrate through the forming gap and stop its development. When such a time-independent solution is no longer possible, planet migration stops and surface density evolves with time to form a gap in the disk.

To study such kinematic wave solutions we take $\partial\sigma/\partial\tau = 0$ in equation (28). Integrating resulting equation with respect to z and using the fact that at infinity $\sigma = 1$, $\partial\sigma/\partial z = 0$ and

planetary torques are absent we obtain that

$$v(\sigma - 1) = \rho_{\Sigma}^{-1} \lambda_{\nu} \frac{\partial \sigma}{\partial z} + \lambda_t \frac{\partial}{\partial z} [\varphi(I(z))], \quad I(z) = \text{sign}(z) \frac{\rho_{\Sigma}^{1/2} \zeta^{5/2}}{2^{1/4}} \int_0^z \frac{|z'|^{3/2}}{\sigma^{1/2}} dz', \quad (37)$$

with v given by (26).

WH89 used the same approach when studying gap formation with instantaneous density wave damping. In their case φ varies on scales $\sim h_p$ (wake generation requires a couple of disk scale lengths to complete), i.e. $d\varphi/dz \sim x_{sh}$ in our notation. Also, to significantly affect the migration speed one only needs surface density variations $\Delta\sigma \sim h_p/r_p$ to be produced. Combined with the definition (31) this gives $\sim (h_p/r_p)M_f$ as a value of their critical mass, in agreement with equation (1). In our case this mass is larger because wave damping occurs at greater distance from the planet.

We now explore separately cases of purely inviscid disks and disks with nonzero viscosity.

3.1. Discs with $\nu = 0$

If the viscosity in the disk is absent equation (37) simplifies to

$$\sigma^{1/2}(\sigma - 1) = \text{sign}(z) \frac{\rho_{\Sigma}^{1/2} \zeta^{5/2}}{2^{1/4} v} \lambda_t |z|^{3/2} \varphi'(I(z)), \quad (38)$$

where $\varphi'(I) \equiv d\varphi(I)/dI$.

It follows from the properties of $\varphi'(t)$ that $\sigma = 1$ for $t < t_{sh}$, i.e. for $|z| < 1$, meaning that $\rho_{\Sigma} = 1$ (see Appendix A). This implies that in inviscid disk nothing happens with the disk fluid until tidal wave shocks. After that, for $|z| > 1$, $\varphi' < 0$ suggesting that for inward migration $\sigma > 1$ if $z < 0$ (inner disk) and $\sigma < 1$ if $z > 0$ (outer disk) just like we mentioned before. The effect of this is to slow down the migration. We can rewrite (26) as

$$v = 1 - \lambda_s \left(\int_{-\infty}^{-1} - \int_1^{\infty} \right) \frac{\sigma - 1}{z^4} dz. \quad (39)$$

Since integrals are $O(1)$, one needs $\lambda_s \gtrsim 1$ for the drift velocity to be significantly affected. The solution of the system (38)-(39) depends on 2 parameters only, λ_t and λ_s .

In Fig. 1 we plot the surface density profiles in the vicinity of the planet for several values of λ_t and λ_s . Surface density increases in front of the migrating planet because there are two fluxes of the disk fluid there — one caused by the migration, another by the planetary repulsion — and they are converging causing a surface density pileup. Behind the moving planet these fluxes are in the same direction — away from the planet. For large enough planetary mass repulsion removes

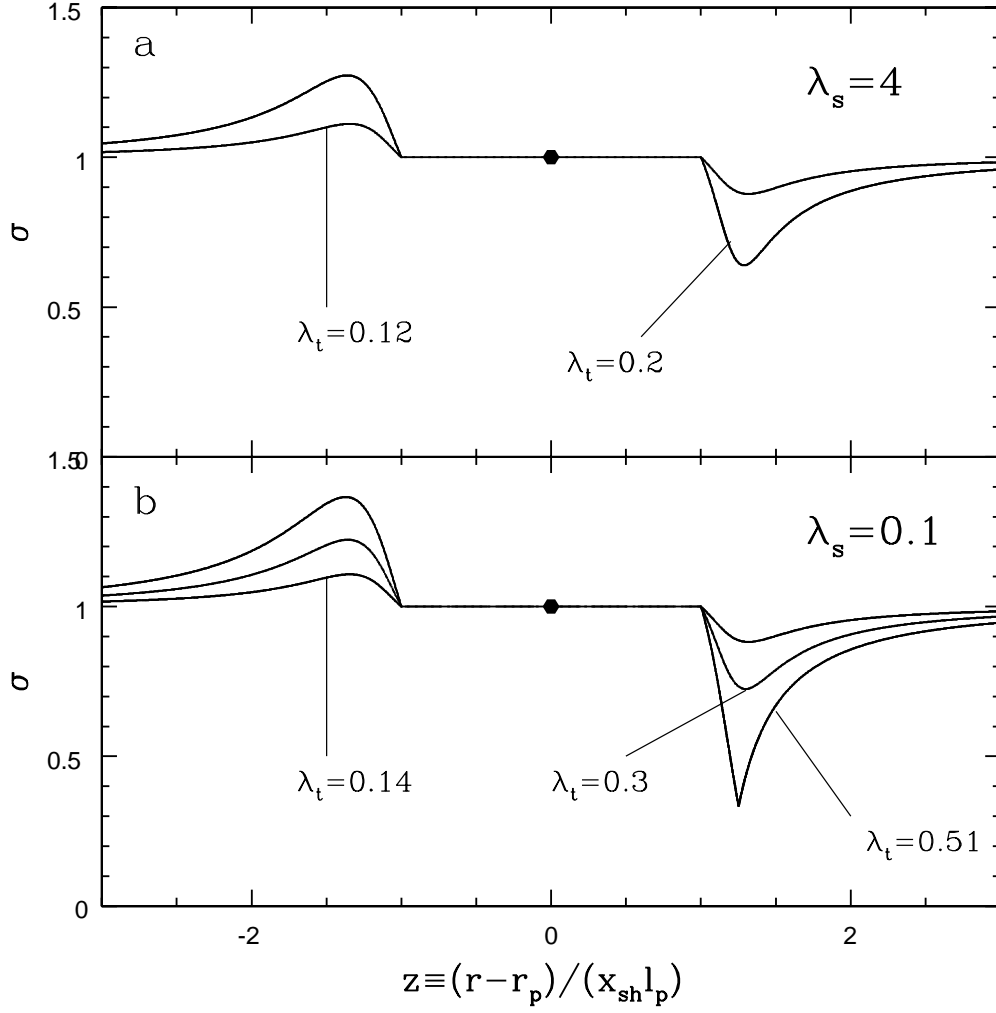


Fig. 1.— Surface density profiles for several values of λ_t and λ_s characterizing the strength of the planet’s gravity and migration feedback. Surface density normalized by its value at infinity is plotted as a function of the distance from the perturber (normalized by the shocking distance $x_{sh} l_p$). Black dot denotes the position of the planet; central star and direction of migration are to the left.

matter from the the disk behind the planet so rapidly that a corresponding surface density decrease cannot be replenished by the flux due to the planetary migration (this flux grows with M_p slower than the one due to the repulsion). As a result a gap is carved out in the disk behind the moving planet.

To describe this process quantitatively let us notice that r.h.s. of equation (38) depends on σ only through an integral over z . For $z > 0$ r.h.s. of (38) varies from

$$f_{min} = \varphi'_{min} \frac{\zeta^{5/2}}{2^{1/4} v} \lambda_t z_{min}^{3/2} \quad (40)$$

to 0, where z_{min} is the value of z for which $\varphi'(I(z))$ reaches minimum, and $\varphi'_{min} = -0.273$ as described in Appendix A. At the same time l.h.s. of (38) varies from $-2 \times 3^{-3/2}$ to 0 (it is minimized at $t = 1/3$). Thus, the solution of (38) does not exist if $|f_{min}| > 2 \times 3^{-3/2}$ or if

$$\lambda_t > \frac{2}{3^{3/2}} \frac{2^{1/4} v}{\zeta^{5/2} |\varphi'_{min}|} z_{min}^{-3/2}. \quad (41)$$

Closer inspection of equation (38) reveals that σ develops an infinite derivative when this happens. This would lead to the violation of the Rayleigh's criterion and instability associated with this will presumably help to clear out a gap on the outer side of the disk.

Once the gap is formed, density waves launched in the outer disk are not able to propagate to infinity and to transfer all their angular momentum to the disk material. Instead they reflect from the edge of a forming gap and return to the planet. Here they could interact gravitationally with planet thus canceling part (or all) of the wave angular momentum launched in the outer direction. It would diminish the influence of the outer disk on the migration and planetary drift will stall. For this to occur planet needs to absorb only a small fraction of one-sided angular momentum F_0 , about h_p/r_p , which is obvious from the discussion in §2.3 and Appendix B.

For $\lambda_s \lesssim 1$ it turns out that $z_{min} \approx 1.2$ and $v \approx 1$ (planet is still rapidly migrating, feedback effects are not strong enough to slow it down) when steady-state solution for σ is no longer possible. This means that gap opens when

$$\lambda_t \gtrsim 0.5 \quad \text{if} \quad \lambda_s \lesssim 1. \quad (42)$$

This corresponds to planetary mass

$$M_p \gtrsim M_t \approx 2.3 Q^{-5/7} M_1 \quad \text{if} \quad \lambda_s \lesssim 1. \quad (43)$$

For stable disks ($Q \gg 1$) this criterion gives quite low mass at which gap forms although larger by $\sim (r_p/h_p)Q^{2/7}$ than the one suggested by Ward & Hourigan (WH89).

For $\lambda_s \gtrsim 1$ the situation is somewhat different. Here the feedback from the density perturbations to the drift velocity of the planet becomes important. To study this region of the parameter space let us divide equation (38) by $z^4 \sigma^{1/2}$ and integrate the result from $-\infty$ to -1 and from 1 to

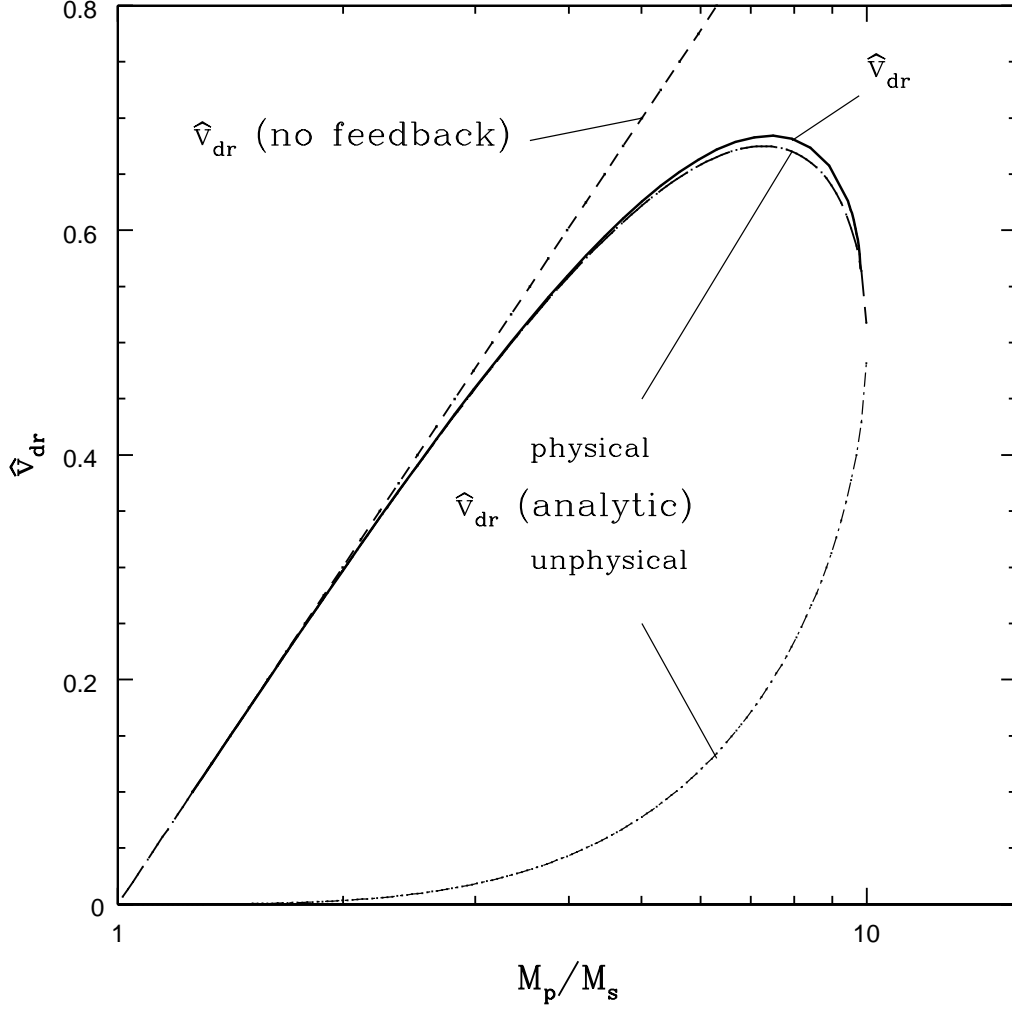


Fig. 2.— Drift velocity of the planet of mass M_p with feedback \hat{v}_{dr} , normalized by the drift velocity of the body with mass M_s [see equation (47)] without feedback, as a function of M_p/M_s . Velocity \hat{v}_{dr} is shown by the thick solid line. Also shown are the drift velocity without feedback and analytical approximation for \hat{v}_{dr} given by equations (44) & (45) (both physical and unphysical roots are displayed).

∞ , thus forming integrals present in r.h.s. of equation (39). Substituting them into (39) we find that

$$v(1-v) = \lambda_s \lambda_t \varrho(\lambda_s, \lambda_t), \quad \text{where} \quad \varrho(\lambda_s, \lambda_t) = -\frac{\zeta^{5/2}}{2^{1/4}} \left(\int_{-\infty}^{-1} + \int_1^{\infty} \right) \frac{\varphi'(I(z))}{\sigma^{1/2}|z|^{5/2}} dz. \quad (44)$$

It turns out that function $\varrho(\lambda_s, \lambda_t)$ depends on λ_s, λ_t only very weakly [which is reasonable because r.h.s. of (44) does not depend on the rapidly varying factor $\sigma - 1$]. We confirmed this numerically and found that

$$\varrho(\lambda_s, \lambda_t) \approx 0.31. \quad (45)$$

Since l.h.s. of equation (44) attains its maximum equal to $1/4$ at $v = 1/2$, no solution of equation (44) for v exists when

$$\lambda_s \lambda_t > \frac{1}{4\varrho(\lambda_s, \lambda_t)} \quad \text{if} \quad \lambda_s \gtrsim 1. \quad (46)$$

From the condition (46) and definitions (27) & (31) we can find that the feedback stops migration when

$$M_p > M_s \approx 5.8 \left(Q^{-1} \frac{h_p}{r_p} \right)^{5/13} M_1 \quad \text{if} \quad \lambda_s \gtrsim 1. \quad (47)$$

In Fig. 2 we show the dependence of the drift velocity on the mass of a perturber in the regime of strong migration feedback. Drift velocity is normalized by the drift speed which a body of mass M_s [equation (47)] would have if the feedback were absent (straight dashed line on this Figure). Analytical solution of equation (44) is also displayed and is in good agreement with numerical result. One can easily see that $d\hat{v}_{dr}/dM_p = -\infty$ when $M_p = M_s$. This situation is analogous to what WH89 have found in their inertial limit calculations in the case of instantaneous damping. When a solution for the drift velocity does not exist any more planetary torques cannot support a constant flux of disk material seen in planet's frame and time dependent evolution commences leading to a gap formation and stalling the migration. Lin & Papaloizou (1986) and WH89 followed this process with time-dependent one-dimensional numerical calculations.

The behavior of the drift velocity correction factor v is shown in Fig. 3. One can see the cutoff of the possible kinematic wave solutions caused by the migration feedback [predicted by equation (46)] on top of the plot where v rapidly diminishes. For small λ_s the criterion given by (42) becomes important. The agreement between our simple analytical considerations and numerical results is very good.

We can unite the gap formation conditions (43) and (47) in a single criterion on the limiting mass of the perturber:

$$\frac{M_p}{M_1} > \min \left[2.3 Q^{-5/7}, 5.8 \left(Q^{-1} \frac{h_p}{r_p} \right)^{5/13} \right]. \quad (48)$$

Because $Q \gg 1$ we can immediately see from equation (48) that gap could be formed by a planet with a mass significantly smaller than M_1 .

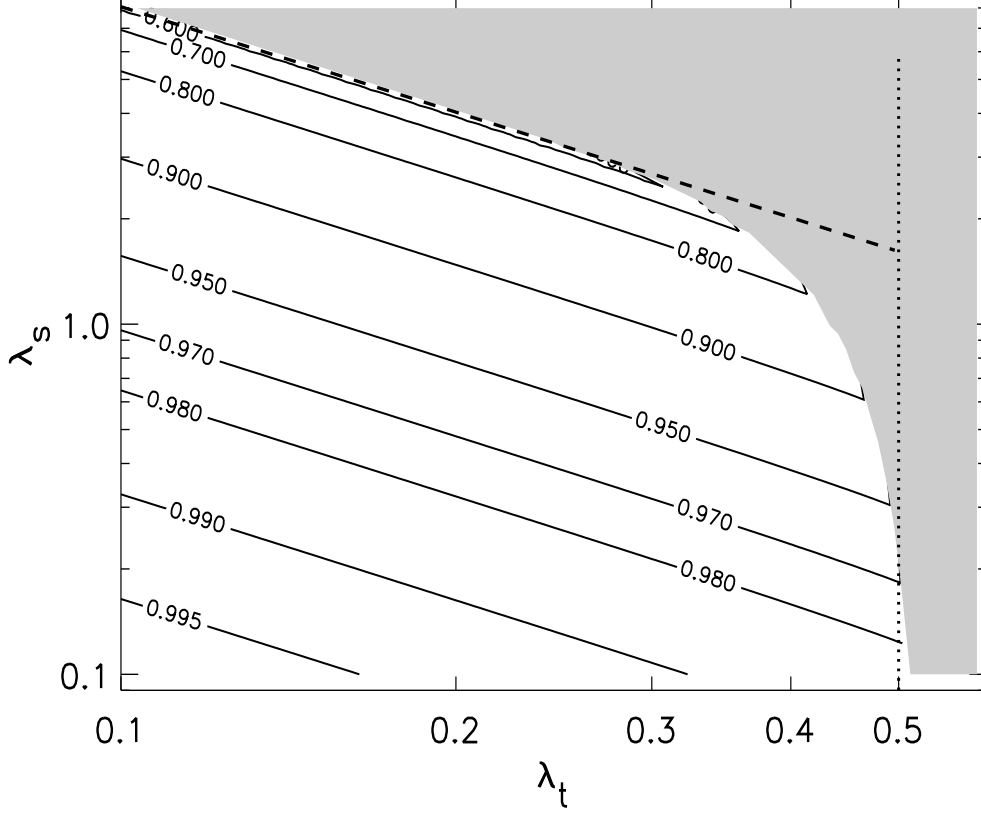


Fig. 3.— Contour plot of the correction factor v for the drift velocity of the planet [see equation (39)] , as a function of parameters λ_t and λ_s . Shaded region corresponds to the part of the parameter space where kinematic wave solution is not possible, migration stalls and gap opens. Vertical dotted line shows the restriction given by equation (42). Dashed line displays the limitation given by equation (46), important when the feedback is strong. One can see a very good agreement between the simple prescription given by equation (48) and actual numerical calculation.

Since our analysis has only exploited the most basic features of the damping function $\varphi(t)$ which are well reproduced by our simple fitting expression (A1) we conclude that the results obtained here would be basically the same if one were to use the exact damping function calculated in GR01.

3.2. Viscous discs.

In the case of nonzero background viscosity present in the disk one faces a more complicated situation than in inviscid disk. Now the solution of kinematic wave equation (37) depends on 4 parameters: λ_t , λ_s , which we had before, λ_ν , which determines the effect of the viscosity on the solution, and x_{sh} , which affects drift velocity feedback. Parameter x_{sh} did not appear in the inviscid case because σ was exactly 1 near the planet in that case. Viscosity changes this picture because now σ varies near the planet, and this is important for the feedback, because tidal forcing is very strong in the immediate vicinity of the planet [see equation (B6)].

In the case $\nu \neq 0$ one can integrate equation (37) once to obtain

$$\sigma - 1 = \rho_\Sigma \frac{\lambda_t}{\lambda_\nu} \left\{ \frac{v\rho_\Sigma}{\lambda_t} \int_z^\infty \exp \left[-\frac{v\rho_\Sigma}{\lambda_\nu} (z - z') \right] \varphi(I(z')) dz' - \varphi(I(z)) \right\}, \quad (49)$$

or

$$\sigma - 1 = \rho_\Sigma \frac{\lambda_t}{\lambda_\nu} \int_0^\infty \left[\varphi \left(I \left(z + s \frac{\lambda_\nu}{v\rho_\Sigma} \right) \right) - \varphi(I(z)) \right] \exp(-s) ds. \quad (50)$$

If one integrates r.h.s. of (50) by parts and takes the limit $\lambda_\nu \rightarrow 0$, equation (38) is easily recovered.

As we have mentioned before, factor ρ_Σ in equation (50) is not equal to unity any more. It should be determined from the self-consistent solution of equation (50) instead. To solve this equation we employ an iterative technique: at each step solution obtained from the previous iteration is substituted into the r.h.s. of (50) to obtain a better approximation until this process converges. As a first trial we use $\sigma(z) = 1$ and $\rho_\Sigma = 1$.

In Fig. 4a we display several surface density profiles for different λ_ν and fixed $\lambda_t = 0.52$ (gap-forming value of λ_t in the inviscid disk without the migration feedback) and $\lambda_s = 0$ (thus assuming that migration feedback is absent). For small λ_ν profile is almost the same as in the inviscid disk; one can show that $(\sigma - 1) \sim \exp(-\lambda_\nu^{-1})$ for small z . As λ_ν grows and becomes comparable with λ_t perturbations of σ near $z = 0$ become significant which has profound implications for the migration feedback if $\lambda_s \neq 0$. However, as $\lambda_\nu \gtrsim \lambda_t$ viscous diffusion becomes so strong that surface density inhomogeneities in the vicinity of the planet become very small (as in the case $\lambda_\nu = 3$ in Fig. 4a).

It turns out that in the viscous case (exactly like in the inviscid one) strong tidal perturbations can open a gap in the disk even without the drift velocity feedback. The presence of the viscosity

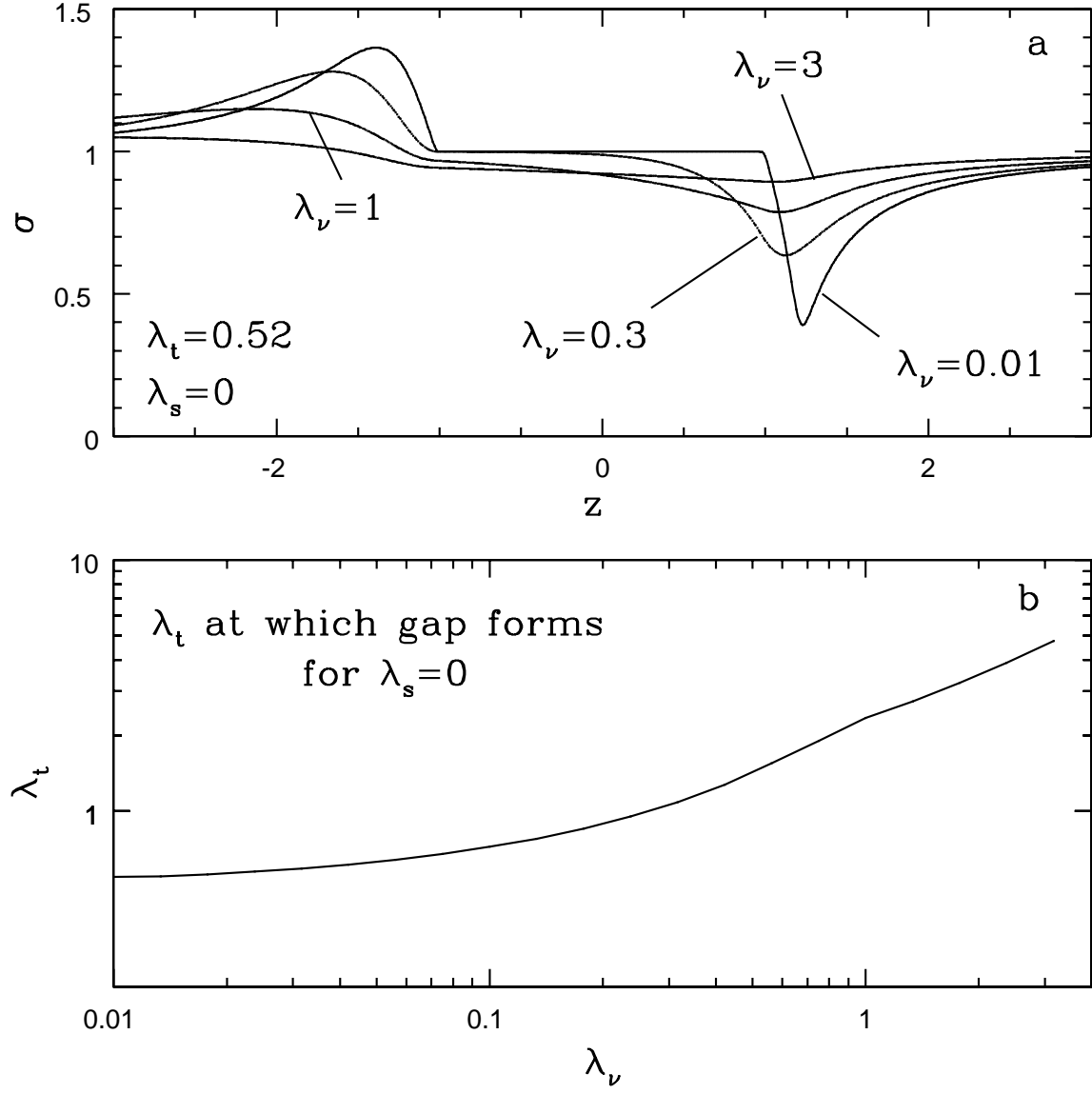


Fig. 4.— (*top*) Plot of the surface density as a function of distance from the perturber for different values of viscous parameter λ_ν : 0.01, 0.3, 1, 3. One can see how the viscosity smooths out density inhomogeneities in the disk and produces variations of σ in the region $|z| \lesssim 1$, leading to a stronger drift velocity feedback. (*bottom*) Maximum possible λ_t for which planet is able to migrate without opening a gap as a function of λ_ν . It is assumed that feedback is absent: $\lambda_s = 0$. One can see that viscosity tends to inhibit gap formation.

only changes the threshold value of the parameter λ_t so that it increases when λ_ν (and, correspondingly, viscosity) grows. The dependence of this limiting λ_t on λ_ν for $\lambda_s = 0$ (no feedback) is shown on Fig. 4b and is easy to understand: to open a gap the planet has to overcome not only the drift of the fresh disk material into a forming gap behind the planet but also the diffusion due to the viscous stresses which tends to smooth any inhomogeneities in σ and fill the gap. Thus, larger λ_ν require larger λ_t to open a gap.

When $\lambda_s > 0$ drift velocity feedback from the surface density perturbations could become important before λ_t reaches the threshold value mentioned above. In Fig. 5 we show the boundary of the region in the $\lambda_t - \lambda_s$ parameter space where a planet could migrate without opening a gap for several values of λ_ν . As we have said before, for small λ_s region where the gap cannot be cleared expands as λ_ν increases confirming results for $\lambda_s = 0$ displayed in Fig. 4b. For larger λ_s , however, the situation is more complicated: for a fixed strength of planetary torques (characterized by λ_t) this region first rapidly contracts but then expands again with increasing λ_ν . It could be explained as follows: when $\lambda_\nu \ll \lambda_t$ viscosity cannot strongly change the surface density near the planet, $\rho_\Sigma \approx 1$ and drift velocity feedback is only slightly stronger than in the inviscid case. When viscosity grows so that $\lambda_\nu \sim \lambda_t$, surface density perturbations are strong and viscosity is significant to sufficiently modify surface density in the planetary vicinity leading to a strong feedback: as it follows from equations (B6) and (26) even small deviations of σ from $\sigma(0) = \rho_\Sigma$ are strongly amplified by a factor $\sim z^{-4}$ (and $z \sim 1/x_{sh} \ll 1$) in the expression for the feedback correction. This effect rapidly lowers the upper boundary of the region where planet could migrate and not open a gap. However, when $\lambda_\nu \gg \lambda_t$ strong viscosity smooths out all the inhomogeneities in σ which planet tends to produce and this reduces the role of the feedback — for a fixed λ_t critical λ_s increases. Thus, growing viscosity tends to amplify migration feedback but then attenuates it as λ_ν exceeds λ_t .

In the viscous disk the absence of the steady-state solution of equation (50) caused by the feedback is not enough to open a gap, a fact emphasized in Lin & Papaloizou (1986) and WH89. Planetary torques must also exceed viscous diffusion for this to happen. This additional requirement could be roughly described by a condition that

$$\lambda_t \gtrsim \lambda_\nu \quad \text{or} \quad \frac{r_p}{h_p} \lesssim \frac{0.133}{\alpha} \left(\frac{M_p}{M_1} \right)^2, \quad (51)$$

where we have used equations (32)-(33). This is equivalent to saying that the time to open a gap by tidal torques must be shorter than the viscous diffusion time (see §2.3). The condition (51) coincides with the corresponding criterion used by Lin & Papaloizou (1986) and WH89, but is different from the condition found in GT80 because only gap formation by a torque exerted at a single Lindblad resonance was considered in GT80. In Fig. 6 we plot the boundary of allowed region in the space of physical parameters: M_p/M_1 and r_p/h_p for different values of dimensionless viscosity α and Toomre stability parameter Q . In depicting the boundary curves we have taken into account the condition (51) (straight line parts of the curves are due to this restriction).

For a fixed viscosity, the planet can open a gap only if it is located to the right of the cor-

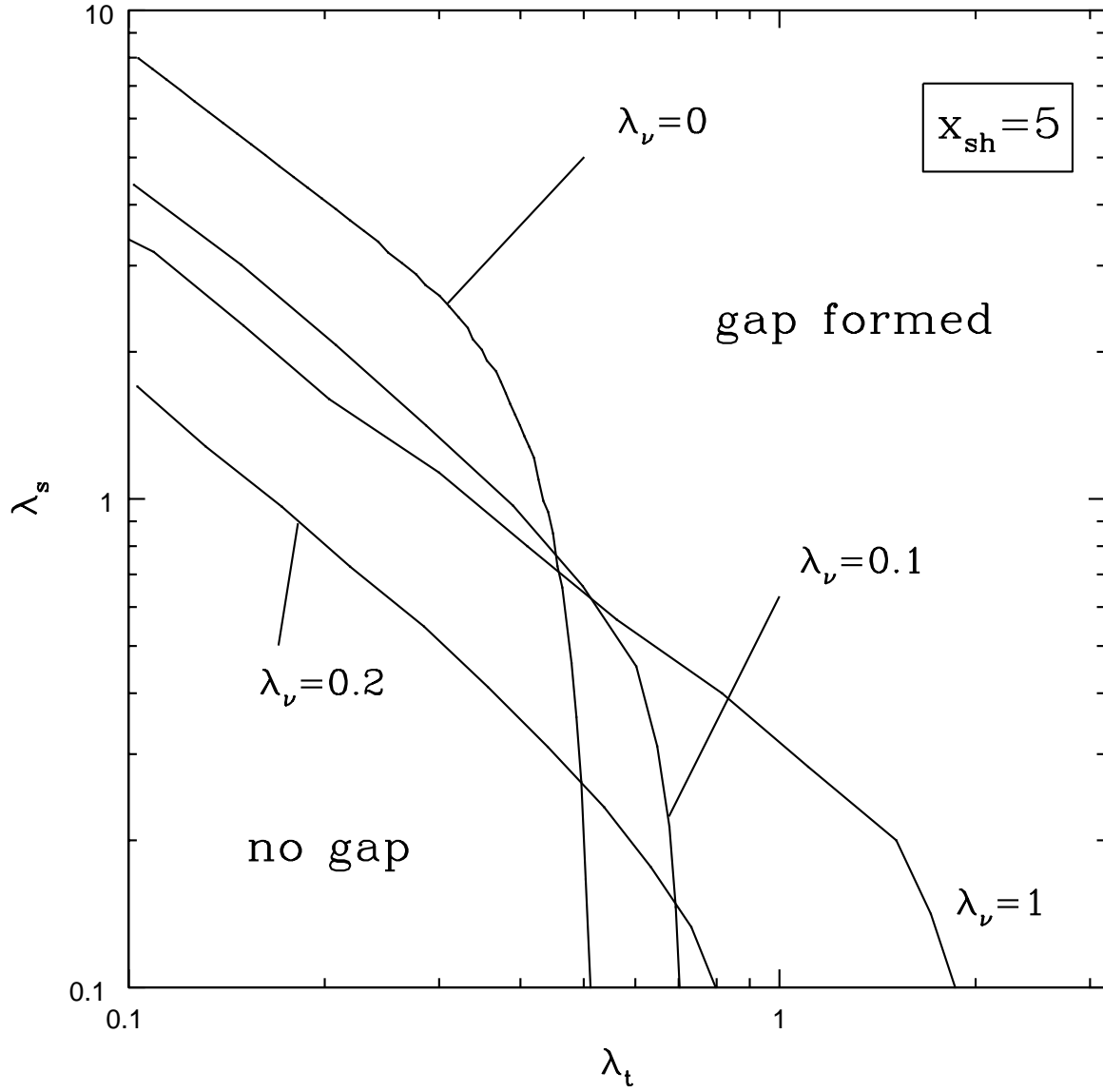


Fig. 5.— Boundary of the region in which planet is able to migrate without opening a gap for different values of the viscous parameter λ_ν in the $\lambda_t - \lambda_s$ plane. Planet migrates when λ_t and λ_s are small and gap is opened for large λ_t and λ_s . It is assumed that $x_{sh} = 5$ is constant for all the planets (meaning that M_p/M_1 is fixed).

responding curve in Fig. 6. For small r_p/h_p (thick disks, small migration feedback) critical M_p is always larger than in the inviscid case and grows with increasing α . As r_p/h_p increases both λ_ν and λ_ν grow according to equations (27) & (30). This rapidly increases the role of the migration feedback so that corresponding limiting M_p could become smaller than in the inviscid case. Complicated shape of the boundary curves for different values of α is due to the before mentioned complexity of the influence of the disk viscosity on the migration feedback.

One can see from the Fig. 6 that for values of r_p/h_p typical in protoplanetary disks ($r_p/h_p \sim 20-30$) $M_p \ll M_1$ if α is small (typically when $\alpha < 10^{-4}$). One can also notice that it is a migration feedback which initiates the evolution of the surface density leading to the gap formation once the condition (51) is fulfilled.

Our consideration of the viscous case assumes that the only surface density gradients producing viscous flux are due to the planet-driven perturbations. However, background surface density varying on scales $\sim r_p$ also contributes to the viscous fluid flux. We neglected this effect completely when we employed our “quasilocal approximation” and omitted factors Σ/r leaving only $d\Sigma/dr$ in our equations (see §2.2). If we were to include these factors in our analysis a picture similar to that described in Ward (1997a) would be found: for large viscosity there would not be an infinite derivative of the drift velocity with respect to M_p at some point as in the case shown in Fig. 2. Instead v_{dr} would smoothly (but rapidly) decrease at some point as M_p grows and the migration would change gradually from type I to type II (Ward 1997a). We are unable to capture this effect working in the framework of our “quasilocal approximation” but this limitation cannot affect our major conclusions concerning the gap forming planetary mass. Also, when the viscosity is small or when the condition (23) is satisfied we should not worry about this effect at all. Using equations (32)-(33) we find that these additional viscous fluxes could be neglected if $r_p/h_p \gg (M_p/M_1)^{-2/5}$ which is usually the case in realistic situations.

4. Discussion

4.1. Limitations of the analysis.

The validity of our consideration is restricted not only by the requirements (23) and $M_p \ll M_1$ but also by the applicability of our nonlinear wave damping prescription. Its validity was extensively discussed in GR01 and Rafikov (2001) and we will not repeat this discussion here. For small enough viscosity ($\alpha < 10^{-4}$) we found (see Fig. 6) that $M_p/M_1 \sim 0.1$. This means that typically $t_{sh} \sim 10$ and $r_{sh} \approx (2-3) \times h_p$ so that the condition $x_{sh} \gtrsim 1$ is marginally fulfilled (note also that $t_{sh} \gg t_0 = 1.89$ which is the offset value of t characterizing the wake generation region, see GR01). Thus, a wave typically shocks after travelling only several h_p from the planet; assumption $x_{sh} \ll r_p/h_p$ is reasonable if $r_p/h_p \gtrsim (10-20)$ which is the case in MMSN model described by equation (2) ($r_p/h_p \approx 25$ at 1 AU). In disks with higher viscosity, the wave will shock even closer to the planet (see §3.2) meaning that initially wake is not tightly wound, but this can hardly change

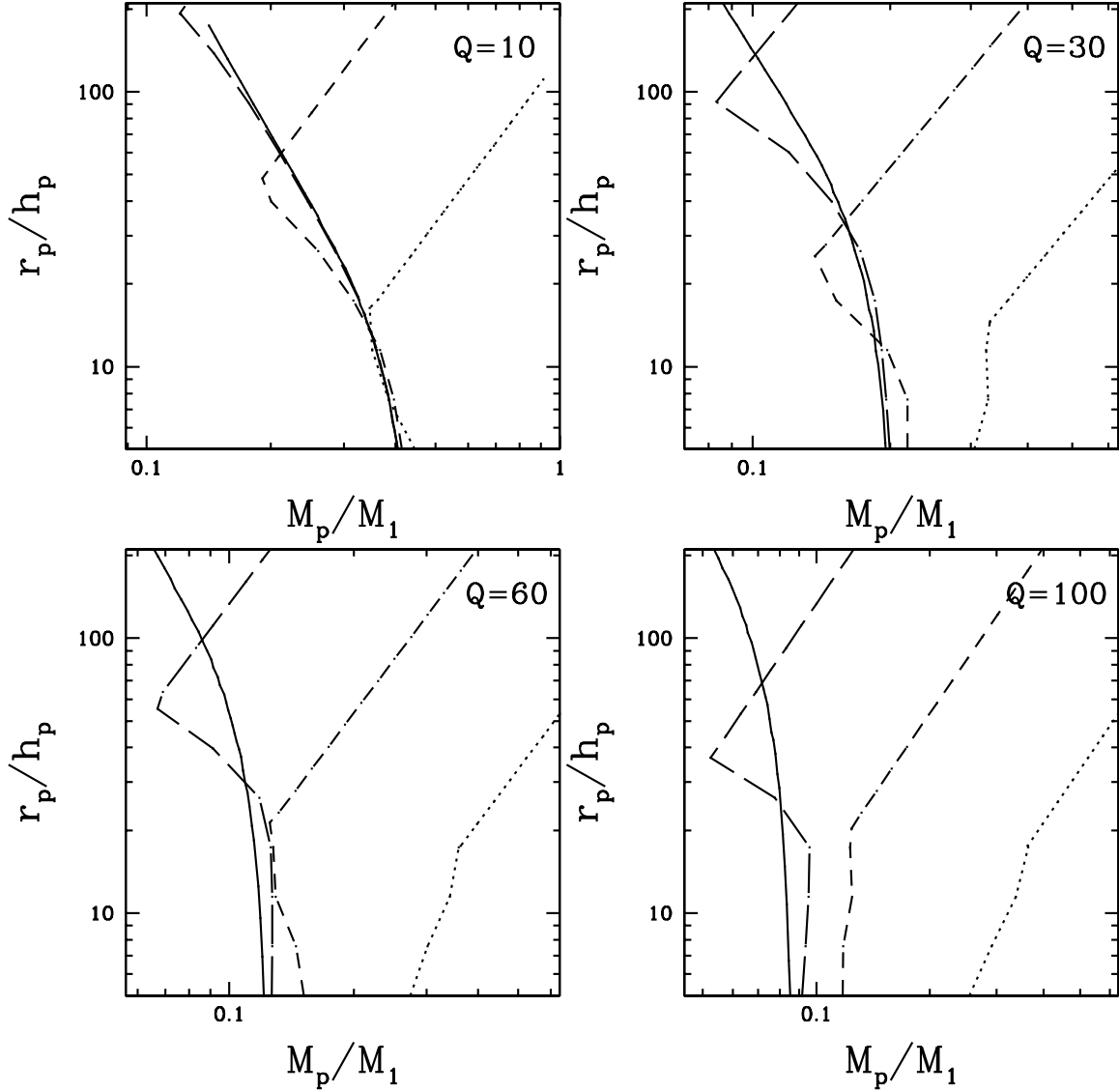


Fig. 6.— Boundary of the region in which planet is able to migrate without opening a gap for different values of dimensionless viscosity α and Toomre stability parameter Q in the $M_p/M_1 - r_p/h_p$ plane. We plot boundaries for $Q = 10, 30, 60, 100$ (labelled on the panels), and $\alpha = 0$ (*solid line*), $\alpha = 10^{-5}$ (*long-dashed line*), $\alpha = 10^{-4}$ (*short-dashed line*), $\alpha = 10^{-3}$ (*dotted line*). Gap in the disk is opened when the system is to the right of the corresponding boundary. Straight line portions of the boundaries represent restriction $\lambda_t > \lambda_v$ necessary for opening a gap in the strong drift velocity feedback limit. Complicated shape of the boundary curves for $\alpha \neq 0$ is due to the complex interaction of the viscosity (affecting the surface density profile near the planet) and the drift velocity feedback.

our gap formation criterion.

4.2. Time-dependent evolution

After the planet grows so massive that a steady-state surface density distribution is no longer possible, time-dependent evolution commences. We do not study this process here. It is clear however that as soon as the gap opening criterion is fulfilled and disk viscosity is overcome by planetary torques, gap clearing starts behind the planet (in the part of the disk opposite to the direction of migration). As we have mentioned before this makes propagation of the density waves in the outer disk impossible, and planet migration which is the result of the small difference of the torques on both sides of the disk, has to stall. When this happens the planet gradually repels the surrounding material, carving out a gap in both parts of the disk. Far from the planet gap expansion can be stopped by the disk viscosity or tidal torques from other planets in the system.

This general picture of gap formation (after the corresponding gap-opening criterion is fulfilled) was confirmed by the one-dimensional time-dependent numerical calculations in Lin & Papaloizou (1986) and WH89. Of course they used a different gap-forming criterion because of their assumed instantaneous damping of the planet-generated density waves. But because the reason for the gap formation in our case is basically the same as in theirs — migration feedback and strong tidal torques exceed the viscous spreading — we expect the time-dependent transition from type I to type II migration in our setting to be similar to what these authors have found in the local damping approximation. The only differences would be the critical mass at which this happens and the transition timescale. Our results for the gap-opening criterion also seem to be in general agreement with the time-dependent two-dimensional simulations carried out by LP93, Bryden et al. (1999), and Nelson et al. (2000), although it is hard to make a direct quantitative comparison because our criterion requires knowledge of a larger number of parameters than is quoted in these studies.

Typical timescale for a gap to form is t_0/λ_t . Using equation (29) or (33), assuming $M_p = 0.1M_1$ and $\lambda_t \sim 1$, one can find that for MMSN parameters this time is about $5 \times 10^4 \Omega^{-1} \approx 7 \times 10^3$ yr at 1 AU and $\sim 2 \times 10^4 \Omega^{-1} \approx 4 \times 10^4$ yr at 5 AU (semimajor axis of Jupiter’s orbit). Such long timescales can put some restrictions on the numerical schemes which would be able to check the validity of our gap-opening criterion (they should also have high spatial resolution, typically a fraction of h_p , and be able to follow the formation and evolution of weak shocks in a disk, although numerical viscosity would be a primary concern).

4.3. Applications

Using the theory developed in §3 we can determine the planetary mass at a particular location in a disk for which gap formation is expected and migration switches to a type II mode. For the MMSN parameters represented by equation (2) one obtains $r_p/h_p \approx 25$, $Q \approx 70$, and $M_1 \approx$

$14 M_{\oplus}$ at 1 AU. Using Fig. 6 we find that for $\alpha \lesssim 10^{-4}$ a gap is opened by a planet with $M_p \approx (0.12 - 0.15)M_1 \approx (1.5 - 2) M_{\oplus}$. Thus, if the Earth was immersed in a gaseous disk at the end of its formation, it probably could not open a gap (nonzero disk viscosity only strengthens this conclusion). Then type I migration would have caused it to drift to the Sun on a timescale $\lesssim 10^5$ yr, but this apparently did not happen. This result supports the usual view according to which the final accumulation of terrestrial planets via the runaway coagulation of planetesimals or giant impacts of Moon-sized planetary embryos occurred *after* the nebula was dispersed in the inner Solar System. Then Earth would not have migrated at all and the question of its survival would not arise.

At 5 AU one finds that $r_p/h_p \approx 16$, $Q \approx 45$, and $M_1 \approx 50 M_{\oplus}$; then critical $M_p/M_1 \approx (0.15 - 0.2)M_1 \approx (7 - 9)M_{\oplus}$ for $\alpha \lesssim 10^{-4}$. Thus, even the usual rocky core of Jupiter alone (without gaseous envelope) would likely be able to open a gap; this would significantly slow down its migration towards the Sun and leave Jupiter on its orbit far from the central star. For some reason this clearly did not happen in systems harboring extrasolar giant planets close to their parent stars, and this issue apparently deserves further study. The value of the critical mass also raises another question: how has Jupiter managed to acquire its huge gaseous mass? It might be that the timescale for the gap formation at the Jupiter’s location is long enough (see §4.2) for the planet to accrete all its mass during the gap-opening stage if the core instability (Mizuno 1980) was operating since the very beginning (core instability and associated planetary mass growth might have actually triggered the gap formation). But obviously more detailed consideration of the gas accretion process is needed to definitely answer this question.

An interesting result following from this analysis is that when a gap is opened in an inviscid disk, there is still some material remaining at the planet’s orbit. This material has the form of a ribbon with radial width equal to $2h_p x_{sh}$ because waves launched by the planet cannot shock prior to travelling a minimum necessary distance, and, thus, cannot transfer their angular momentum to the disk and cause its evolution. The effects of wave action reflection from the edge of the ribbon or the presence of the moderate viscosity in the disk might lead to the dissipation of such a gas torus. But if these effects are not very strong then this ribbon phenomenon might have observable manifestations.

5. Conclusions

Using a realistic damping prescription for tidally-induced density waves we have studied the conditions necessary for a gap to be formed in a gas disk in the vicinity of a planet. It was shown that for small enough planetary mass a steady-state solution for the surface density perturbations exists in the reference frame migrating with the planet. Then the details of the surface density distribution in the disk depend on only 4 parameters: the aspect ratio in the disk h_p/r_p , the viscosity (represented here by its dimensionless analog α), the Toomre stability parameter of the disk Q , and the ratio of the planetary mass to a fiducial mass M_1 defined in equation (3). Only systems with

$\alpha < 10^{-3}$ were considered because larger viscosity could violate our damping prescription based on the nonlinear wave dissipation. We have demonstrated that in disks with Q between ~ 10 and ~ 100 (which should be typical in passive protoplanetary disks irradiated by their central stars) a gap is opened when the planetary mass reaches $(1 - 10)M_{\oplus}$, depending on the disk viscosity and the planet’s location in the nebula. Planets further away from the central star must be more massive to repel the gas in their vicinity. We obtained an analytical criterion for a gap-forming planetary mass in inviscid disks [see equation (48)]. Our result for this critical mass is in between the previous estimates of this quantity obtained by Ward & Hourigan (1989) and Lin & Papaloizou (1993) because of the different density wave damping function (WH89) and moderate requirements for the nonlinearity of the wave (LP93).

The apparatus developed here for following the disk surface density evolution and studying stationary structures in the disk such as the kinematic wave solutions (see §3) could be generalized to other astrophysical problems; for example it could be used to study global disk evolution or to extend our gap formation analysis to a system containing several planets.

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A. Angular momentum flux $\varphi(t)$.

GR01 have calculated the dependence of the angular momentum flux of shock damped density waves using the full numerical solution for the propagation and damping of the wake. Here for simplicity we use a simple matching function which reproduces the main features of the numerical solution:

$$\varphi(t) = \begin{cases} 0, & t < t_{sh}, \\ \left[1 + (t/t_{sh} - 1)^2\right]^{-1/4}, & t > t_{sh}. \end{cases} \quad (\text{A1})$$

The behavior of the true solution and the matching function is shown in Fig. 7. For a planet-generated wake profile $t_{sh} = 0.79$ (neglecting the fact that there is some distance t_0 over which the wake profile forms). Our simple fit never deviates from the true $\varphi(t)$ by more than 17%. Asymptotically $\varphi(t) \propto t^{-1/2}$ for $t \rightarrow \infty$, as in the numerical solution. Also $d\varphi(t)/dt = 0$ when $t = t_{sh}$.

To study the surface density profile near the planet in inviscid disks it is important to know the behavior of $\varphi'(t) = d\varphi(t)/dt$. For our assumed matching function (A1) one can find that

$$\varphi'(t) \propto (t_{sh} - t), \quad t - t_{sh} \ll 1, \quad \varphi'(t) \propto -t^{-3/2}, \quad t \gg 1, \quad \text{and} \quad \varphi'(t) = 0 \text{ for } t < t_{sh}. \quad (\text{A2})$$

The function $\varphi'(t)$ reaches its minimum value equal to $\varphi'_{min} = -0.273$ at $t_{min} = 1.43$.

B. Drift velocity

From the global conservation of the angular momentum we find the velocity of migration driven by the planetary gravitational torques:

$$v_d = -\frac{2}{M_p r_p \Omega_p} \int \frac{\partial F}{\partial r} dr, \quad (\text{B1})$$

where the integral is taken over the whole disk. One can write $dF/dr = \Sigma(r)df/dr$, where F is the angular momentum flux produced by the planet per unit surface density at distance r . GT80 have demonstrated that function df/dr rapidly falls off like $|r - r_p|^{-4}$ for $|r - r_p| \gtrsim h_p$.

Ward (1986) has shown that $df/dr = \text{sign}(r - r_p)df/dr|_0[1 + O(h_p/r_p)(r - r_p)/h_p]$, where $df/dr|_0$ is an *even* function of $r - r_p$ and terms of order h_p/r_p arise because of the asymmetries in the torque generation pattern intrinsic for Keplerian disks. The surface density itself may vary on different scales. In our particular case we are interested in the variations on scales of order r_p [represented by $\Sigma_0(r)$] which lead to migration in the first place, and variations on scales $\sim h_p x_{sh} \ll r_p$, which result from the planet-driven surface density evolution. In the vicinity of the planet Σ could be written in the following form:

$$\Sigma(r) = \Sigma_0(r) \left[1 + \frac{\Sigma - \Sigma_0(r)}{\Sigma_0(r)}\right] \approx \Sigma_p \left[1 + \frac{2h_p}{3r_p} \frac{d \ln \Sigma_0}{d \ln r} x\right] \times \left[1 + \frac{\Sigma - \Sigma_p}{\Sigma_p}\right]. \quad (\text{B2})$$

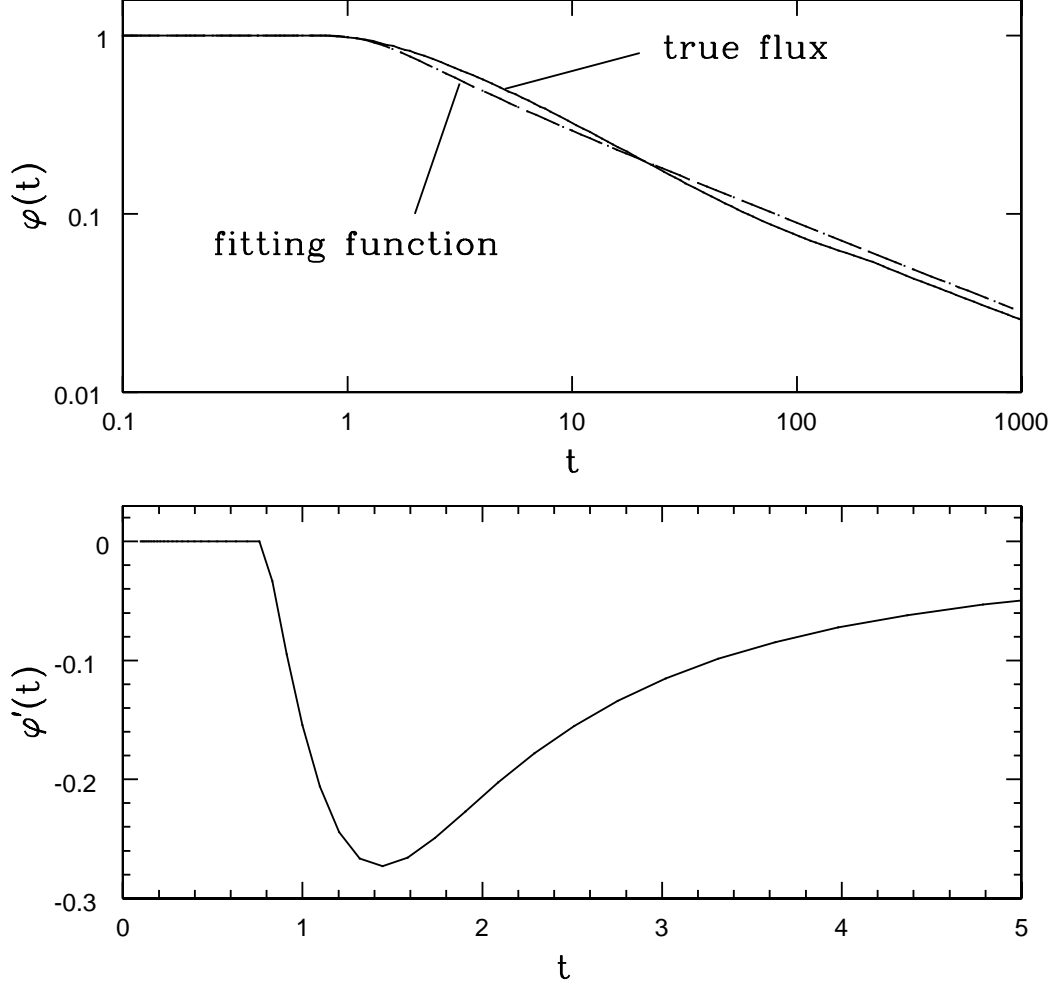


Fig. 7.— (*top*) Behavior of the dimensionless angular momentum flux φ (solid line) and a simple fit (dashed line) given by equation (A1). The flux diminishes as t increases due to shock damping. (*bottom*) Plot of $\varphi'(t)$ calculated using the fitting function (A1).

The replacement of $\Sigma_0(r)$ with Σ_p in the last bracket assumes that variations due to planetary torques are more rapid than background ones (because damping length is supposed to be much shorter than r_p).

Substituting this in (B1) and recalling that $df/dr|_0$ falls off beyond several h_p from the planet one finds that

$$\int \frac{\partial F}{\partial r} dr = F_0 \left[\beta \frac{h_p}{r_p} + \frac{\Sigma_p}{F_0} \int \text{sign}(x) \frac{df}{dx} \Big|_0 (\rho_\Sigma^{-1} \sigma - 1) dx \right]. \quad (\text{B3})$$

Here β is a constant of the order of unity which depends on the global gradients of the surface density $k = -d \ln \Sigma / d \ln r$ and temperature $l = -d \ln T / d \ln r$ in the disk. The first term in brackets arises as a result of the asymmetry in torques produced in the inner and outer parts of the disk. Ward (1986) advocates that for $Q = \infty$ (Q is a Toomre stability parameter)

$$\beta \times 2 \left\{ \frac{4}{9} \mu_{max}^3(Q) [2K_0(2/3) + K_1(2/3)]^2 \right\} = 6.5(1 + 0.06k + 1.2l), \quad (\text{B4})$$

and for $Q = 2$ r.h.s. of (B4) changes to $72(1 - 0.19k + 0.95l)$. It means that the surface density decreasing from the center of the disk leads to a very weak acceleration of drift in $Q = \infty$ disk and slows it down for moderate values of Q . Further we will assume for simplicity that the expression in parentheses in r.h.s. of (B4) is equal to 2 (which corresponds to $\beta \approx 7$).

Using (B3) we obtain that

$$v_d = -\frac{2\beta F_0}{M_p r_p \Omega_p} \frac{h_p}{r_p} v, \quad \text{where} \quad v = 1 + \frac{1}{\beta} \frac{r_p}{h_p} \frac{\Sigma_p}{F_0} \int \text{sign}(x) \frac{df}{dx} \Big|_0 (\rho_\Sigma^{-1} \sigma - 1) dx \quad (\text{B5})$$

is a correction factor for the drift velocity caused by the feedback from the surface density variations to the planetary migration.

For large $x = (r - r_p)/l_p$ it could be demonstrated (GT80) that

$$\frac{df}{dx} \Big|_0 = \frac{F_0}{\Sigma_p} \frac{3}{\mu_{max}^3 x^4}. \quad (\text{B6})$$

For $x \lesssim 1$ torque rapidly decreases (“torque cutoff”). For simplicity we will assume that the dependence given by equation (B6) holds true even for $x \sim 1$, then we only need to correct the cutoff value of x so that the integral of (B6) gives us the right amount of the angular momentum flux F_0 at infinity. It is done by setting $df/dx|_0 = 0$ for $|x| < \mu_{max}^{-1}$; for $|x| > \mu_{max}^{-1}$ we assume $df/dx|_0$ to be given by equation (B6). Using equation (B5) we finally obtain equation (26) and (27).

Ideally, one should also take into account gradient of Σ on short damping scale in the expression for df/dr as it was done by Ward (1986). This could produce some modification because surface density gradients displace the positions of the Lindblad resonances and modify the torque cutoff. However, while variations of Σ produce contribution to the drift $\propto \Delta\sigma$, presence of the gradients of Σ gives rise to the effects of order $d\sigma/dx \sim \Delta\sigma/x_{sh}$. This implies that contribution due to the gradient of the surface density is $\sim 1/x_{sh}$ compared with the last term in (B3) and is unimportant for $M_p \ll M_1$.